

# Bell's inequality, the Pauli exclusion principle and baryonic structure

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## Abstract

Bell's inequality has been traditionally used to explore the relationship between hidden variables and the Copenhagen interpretation of quantum mechanics. In this paper, another use is found. Bell's inequality is used to derive a coupling principle for elementary particles and to give a deeper understanding of baryonic structure. We also give a derivation of the Pauli exclusion principle from the coupling principle. Pacs: 14.20-c, 12.90+b, 3.65.Bz, 12.40.Ee

## 1 Introduction

Bell's Inequality [1] was derived by John Bell in 1964 as a response to The Einstein-Podolsky-Rosen Paradox [2], a problem pertaining to the foundations of quantum physics. Bell saw his inequality as being able to discern between two different epistemological views of quantum mechanics, the one proposed by EPR and the one proposed by the Copenhagen interpretation of quantum theory.

In this paper we point out another implication of Bell's work. We first derive a coupling principle directly from the inequality and show that the Pauli principle can be viewed as a special case of this coupling. We then apply the principle to further our understanding of baryonic structure and note that the case of spin  $\frac{3}{2}$  baryons can be analyzed in one of two ways as reflected in the following assumptions:

- (1) In every direction the spin will be observed to be  $\pm 3/2$ .
- (2) There exists some direction in which the spin will be observed to be  $\pm 3/2$ .

Assumption (1) in fact is the key point of a previous paper [6] and will not be discussed here. Assumption (2), on the other hand, when combined with the coupling principle mentioned above, enables us to explain the statistical structure of the  $\Delta^{++}$  and the  $\Omega^-$  particles without any recourse to color. It is discussed in section four of the paper.

## 2 A Coupling Principle

Consider three (or more) particles in the same spin state. In other words, if a measurement is made in an arbitrary direction  $a_1$  on ONE of the three particles, then the measurements can be predicted with certainty for the same direction for

each of the other particles. We point out immediately that such spin correlations are isotropic for the particles under discussion and that we are not dealing with a polarization phenomena where spin correlations exist for a preferred direction. In our case, the particles are spin- correlated in all directions at once, as for example in the case of two particles in a singlet state. Hence, the initial direction of measurement is arbitrary. We refer to such particles as isotropically spin-correlated particles.[3]

Specifically, if we denote a spin up state by the ket  $|+>$  and a spin down state by the ket  $|->$  then without loss of generality, we can assume that the three particles have the joint spin state  $|+, +, +>_1$  ( $|-, -, ->_1$ ), where the suffix 1, refers to the observed spin states in the arbitrary direction  $a_1$ .

In the language of probability, we can say that if the spin state of a particle is  $|+>_1$  then the corresponding spin state of each of the other two particles can be predicted (for the same direction) with probability 1. Furthermore, the probability 1 condition means that in principle spin can now be measured simultaneously in the three different directions  $a_1, a_2, a_3$ , for the three particle ensemble (see Fig. 1). Let  $P$  denote the joint probability measure relating the measurements in the three different directions and recall the fact that if spin is observed to be in the  $|+>_1$  state in direction  $a_1$  for one of the particles then the conditional probability of observing  $|+>_2$  or  $|->_2$  in the direction  $a_2$  for a second particle, is given by  $\cos^2(c\theta_{12})$  or  $\sin^2(c\theta_{12})$  respectively, where  $\theta_{12}$  is the angle subtended by  $a_1$  and  $a_2$  and  $c$  is a constant. For the purpose of the argument below, we will work with  $c = 1/2$ . However, the argument can be made to work for any value of  $c$ , and in a particular way can be applied to the spin of a photon, provided  $c = 1$ .

With notation now in place, we adapt an argument of Wigner [4] to show that isotropically spin-correlated particles must occur in pairs. We prove this by contradiction. Specifically, consider three isotropically spin-correlated particles (see Fig. 1), as explained above. It follows from the probability 1 condition, that three spin measurements can be performed, in principle, on the three particle system, in the directions  $a_1, a_2, a_3$ . Let  $(s_1, s_2, s_3)$  represent the observed spin values in the three different directions. Note that  $s_i = \pm$  in the notation developed above which means that there exists only two possible values for each measurement. Hence, for three measurements there are a total of 8 possibilities in total. In particular,

$$\begin{aligned} \{(+, +, -), (+, -, -)\} &\subset \{(+, +, -), (+, -, -), (-, +, -), (+, -, +)\} \\ \Rightarrow P\{(+, +, -), (+, -, -)\} &\leq P\{(+, +, -), (+, -, -), (-, +, -), (+, -, +)\}. \end{aligned}$$

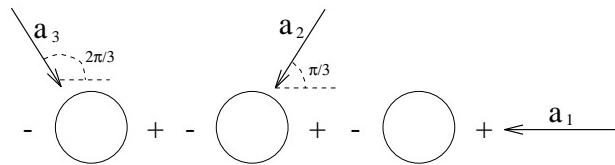


Figure 1: Three isotropically spin-correlated particles.

Therefore,

$$\frac{1}{2} \sin^2 \frac{\theta_{31}}{2} \leq \frac{1}{2} \sin^2 \frac{\theta_{23}}{2} + \frac{1}{2} \sin^2 \frac{\theta_{12}}{2}.$$

If we take  $\theta_{12} = \theta_{23} = \frac{\pi}{3}$  and  $\theta_{31} = \frac{2\pi}{3}$  then this gives  $\frac{1}{2} \geq \frac{3}{4}$  which is clearly a contradiction. In other words, three particles cannot all be in the same spin state with probability 1, or to put it another way, isotropically spin-correlated particles must occur in pairs.

Finally, as noted above, this argument applies also to spin 1 particles, like the photons, provided full angle formulae are used instead of the half-angled formulae.

### 3 Pauli exclusion principle

The above results can be cast into the form of a theorem (already proven above) which will be referred to as the “coupling principle”.

**Theorem 1** (*The Coupling Principle*) *Isotropically spin- correlated particles must occur in PAIRS.*

In practice, isotropically spin-correlated particles occur when the particles’ spin are either anti-parallel (singlet state) or parallel to each other.

We now show that when a system of indistinguishable particles contain “coupled” particles then this system of particles must obey fermi-dirac statistics. We first do this for a 2- particle spin-singlet state system and then extend the result to an n-particle system. Throughout  $\lambda_i = (q_i, s_i)$  will represent the quantum coordinates of particle i, with  $s_i$  referring to the spin coordinate and  $q_i$  representing all other coordinates. In practice,  $\lambda_i = (q_i, s_i)$  will represent the eigenvalues of an operator defined on the Hilbert space  $L^2(\mathcal{R}^3) \otimes H_2$ , where  $H_2$  represents a two-dimensional spin space of particle i. We will mainly work with  $\lambda_i$ . However, occasionally, in the interest of clarity, we will have need to distinguish the  $q_i$  from the  $s_i$ .

**Corollary 1** *Let  $|\psi(\lambda_1, \lambda_2)\rangle$  denote a two particle state where the  $\lambda_1$  and  $\lambda_2$  are as defined above. If the particles are in a spin-singlet state then their joint state function will be given by*

$$|\psi_i(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}} [|\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle - |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle].$$

*In other words, coupled particles obey fermi-dirac statistics.*

**Proof:** The general form of the two particle eigenstate is of the form

$$|\psi(\lambda_1, \lambda_2)\rangle = c_1 |\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle + c_2 |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle.$$

Since the particles are in a spin-singlet state then  $P(\lambda_1 = \lambda_2) \leq P(s_1 = s_2) = 0$ . Therefore,  $\langle \psi(\lambda_1, \lambda_1) | \psi(\lambda_1, \lambda_1) \rangle = 0$  and hence  $|\psi(\lambda_1, \lambda_1)\rangle = 0$ , from the inner

product properties of a Hilbert space. It follows, that  $c_1 = -c_2$  when the particles are coupled and normalizing the wave function gives  $|c_1| = \frac{1}{\sqrt{2}}$ . The result follows. QED

Note that the same result can also be used to describe particles whose spin correlations are parallel to each other in each direction. This can be done by correlating a measurement in direction  $a$  on one particle, with a measurement in direction  $-a$  in the other. In this case, the state vector for the parallel and anti-parallel measurements will be found to be by the above argument:

$$|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}}[|\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2(\pi))\rangle - |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1(\pi))\rangle]$$

where the  $\pi$  expression in the above arguments, refer to the fact that the measurement on particle two is made in the opposite sense, to that of particle one.

This result can now be generalized to derive the Pauli exclusion principle for a system of  $n$  indistinguishable particles containing at least one pair of coupled particles. First, note the following use of notation. Let  $|\psi_i(\lambda_j)\rangle = \psi_i(\lambda_j)\vec{e}$  where  $\psi_i(\lambda_j)$  refers to particle  $i$  in the state  $|\psi_i(\lambda_j)\rangle$  and  $\vec{e}$  is a unit vector. Then

$$\begin{aligned} |\psi_i(\lambda_j)\rangle \otimes |\psi_k(\lambda_l)\rangle &= [\psi_i(\lambda_j)\vec{e}_1] \otimes [\psi_k(\lambda_l)\vec{e}_2] = \psi_i(\lambda_j)\psi_k(\lambda_l)\vec{e}_1 \otimes \vec{e}_2 \\ &= |\psi_k(\lambda_l)\rangle \otimes |\psi_i(\lambda_j)\rangle . \end{aligned}$$

In other words, the tensor product is commutative. From now on we will drop the ket notation and simply write that  $\psi_i(\lambda_j) \otimes \psi_k(\lambda_l) = \psi_k(\lambda_l) \otimes \psi_i(\lambda_j)$ , with ket notation being understood. We also denote an  $n$ -particle state by  $\psi_{1\dots n}[\lambda_1, \dots, \lambda_n]$  where the subscript  $1\dots n$  refer to the  $n$  particles. However, when there is no ambiguity involved we will simply write this  $n$ -particle state as  $\psi[\lambda_1, \dots, \lambda_n]$  with the subscript  $1\dots n$  being understood. Finally, note that for an indistinguishable system of  $n$  particles

$$\psi[\lambda_1, \dots, \lambda_n] = \sum_P \sigma_P c_P \psi(\lambda_1, \dots, \lambda_n)$$

where  $\psi(\lambda_1, \dots, \lambda_n) = \psi_1(\lambda_1) \otimes \dots \otimes \psi_n(\lambda_n)$  and  $\sigma_P(\psi_1 \otimes \dots \otimes \psi_n) = \psi_{i_1} \otimes \dots \otimes \psi_{i_n}$ , gives a permutation of the states. With this notation, we now prove the following theorem:

**Theorem 2 (The Pauli Exclusion Principle)** *A sufficient condition for a system of  $n$  indistinguishable particles to exhibit fermi-dirac statistics is that it contain spin-coupled particles .*

**Proof:** We will work with three particles, leaving the general case for the Appendix. Consider a system of three indistinguishable particles, containing spin-coupled particles. Using the above notation and applying Cor 1 in the second line below, we

can write:

$$\begin{aligned}
\psi[\lambda_1, \lambda_2, \lambda_3] &= \frac{1}{\sqrt{3}} \{ \psi_1(\lambda_3) \otimes \psi_{23}[\lambda_1, \lambda_2] + \psi_2(\lambda_3) \otimes \psi_{31}[\lambda_1, \lambda_2] + \psi_3(\lambda_3) \otimes \psi_{12}[\lambda_1, \lambda_2] \} \\
&= \frac{1}{\sqrt{3!}} \{ \psi_1(\lambda_3) \otimes [\psi_2(\lambda_1) \otimes \psi_3(\lambda_2) - \psi_3(\lambda_1) \otimes \psi_2(\lambda_2)] \\
&\quad + \psi_2(\lambda_3) \otimes [\psi_3(\lambda_1) \otimes \psi_1(\lambda_2) - \psi_1(\lambda_1) \otimes \psi_3(\lambda_2)] \\
&\quad + \psi_3(\lambda_3) \otimes [\psi_1(\lambda_1) \otimes \psi_2(\lambda_2) - \psi_2(\lambda_1) \otimes \psi_1(\lambda_2)] \} \\
&= \sqrt{3!} \psi_1(\lambda_1) \wedge \psi_2(\lambda_2) \wedge \psi_3(\lambda_3)
\end{aligned}$$

where  $\wedge$  represents the wedge product. Thus the wave function for the three indistinguishable particles obeys the fermi-dirac statistics. The n-particle case follows by induction. QED.

Mathematically it is possible to give other reasons why  $P(\lambda_i, \lambda_i) = 0$  (quark “color” being a case in point) In fact, a necessary and sufficient condition can be formulated for fermi-dirac statistics as follows: In a system of n- indistinguishable particles  $\psi[\lambda_1, \dots, \lambda_i, \lambda_i, \dots] = 0$  for the  $i$  and  $j$  states if and only if

$$\psi[\lambda_1, \lambda_2, \dots, \lambda_n] = \sqrt{n!} \psi_1(\lambda_1) \wedge \psi_2(\lambda_2) \wedge \dots \wedge \psi_n(\lambda_n).$$

The sufficient part of the proof will be the same as in Theorem 2 while the necessity part is immediate. However, the significance of Theorem 2 lies in the fact that for spin-type systems, particles may couple and this coupling causes fermi-dirac statistics to occur. Moreover, the coupling would appear to be a more universal explanation of the Pauli exclusion principle than for example “color”. Not only does it explain the statistical structure of the baryons (see below) but it also explains why in chemistry only two electrons share the same orbital and why “pairing” occurs in the theory of superconductivity.[3],[5, p8]

## 4 Baryonic Structure

We now apply the coupling principle to shed further understanding on the standard model interpretation of baryonic structure. However, first we need to agree on the meaning of spin 1/2 and spin 3/2 particles.

We say that a particle has spin 1/2 if an ensemble of such particles decomposes into two spin states when passed through a Stern-Gerlach magnet. The respective spin states will be referred to as spin 1/2 and spin -1/2 and each occur with probability 1/2. We say that a particle has spin 3/2 if an ensemble of such particles decomposes into four spin states when passed through a Stern-Gerlach magnet, states which correspond to spin 3/2, 1/2, -1/2, -3/2. Moreover, this in effect means that there exists some direction in which the spin will be  $\pm 3/2$  and is consistent with assumption (2) of the introduction.

Next, we apply the coupling principle to spin 1/2 baryons. We note that if two quarks are coupled in a singlet state then the remaining quark is statistically independent of the other two and can exhibit spin 1/2 or spin -1/2 properties. It follows that the spin 1/2 baryon can be explained by the coupling principle if two of the three particles are in a singlet state. Moreover, if in addition the three quarks are indistinguishable then it follows from the Pauli principle (Theorem 2) that the wave function can be expressed as:

$$\psi[\lambda_1, \lambda_2, \lambda_3] = \frac{1}{\sqrt{3}}[\psi_{12}[\lambda_1, \lambda_2]\psi_3(\lambda_3) + \psi_{31}[\lambda_1, \lambda_2]\psi_2(\lambda_3) + \psi_{23}[\lambda_1, \lambda_2]\psi_1(\lambda_3)]$$

where  $\psi_{ij}[\lambda_1, \lambda_2] = \frac{1}{\sqrt{2}}(\psi_i(\lambda_1)\psi_j(\lambda_2) - \psi_i(\lambda_2)\psi_j(\lambda_1))$ . In other words, fermi-dirac statistics results.

The case of spin 3/2 baryons can be explained in a similar way. In this case, it is sufficient to choose the coupled particles in the parallel-correlated-spin state (spin +1 or spin -1). The third particle will have a spin statistically independent of the coupled particles. Putting the three particles together now gives the baryon of spin 3/2, in the sense above. Its state can be formally written as:

$$c(\psi(\lambda_1, \lambda_2)_{12}\psi_3(\lambda_3) + \psi(\lambda_1, \lambda_2)_{31}\psi_2(\lambda_3) + \psi_{23}(\lambda_1, \lambda_2)\psi_1(\lambda_3))$$

where  $\psi_{ij}(\lambda_1, \lambda_2) = \psi_i(\lambda_1)\psi_j(\lambda_2) + \psi_i(\lambda_2)\psi_j(\lambda_1)$ . and  $c = 1/3$  if  $q_1 = q_2 = q_3$ ,  $c = 0$  if  $\lambda_1 = \lambda_2$ ,  $c = \frac{1}{\sqrt{3!}}$  otherwise.

A couple of things to note:

- (1) The fact that two of the quarks are coupled and the other is independent might suggest that the particles can be identified. However, indistinguishability in effect prohibits this. An analogous situation occurs in chemistry when molecules in a liquid form a dynamic equilibrium, and give rise to a constant association and disassociation of the ions, in accordance with indistinguishability. This suggests that the quarks are in effect in dynamic equilibrium with each other, with the coupling being continuously broken and then reformed among the different quarks.
- (2) If it is assumed that only singlet state coupling (in the sense of Theorem 1) is stable then all spin 3/2 baryons will necessarily decompose. This conjecture would seem to be substantiated by the very rapid decay rates observed in ALL spin-3/2 baryons.
- (3) It is impossible for all three particles to be isotropically spin-correlated.
- (4) By using the coupling principle, the principle of isotropy still applies. The particle does not have a preferred orientation in space.
- (5) In the case of three statistically independent particles, the joint state will be similar in form to the expression for parallel coupling. However, the constant coefficients will be different and the state will have the form:

$$c(\psi(\lambda_1, \lambda_2)_{12}\psi_3(\lambda_3) + \psi(\lambda_1, \lambda_2)_{31}\psi_2(\lambda_3) + \psi_{23}(\lambda_1, \lambda_2)\psi_1(\lambda_3))$$

where  $\psi_{ij}(\lambda_1, \lambda_2) = \psi_i(\lambda_1)\psi_j(\lambda_2) + \psi_i(\lambda_2)\psi_j(\lambda_1)$ , and  $c = \frac{1}{\sqrt{8}}$  if  $q_1 = q_2 = q_3$ , and  $c = \sqrt{\frac{3}{8}}$  otherwise.

## 5 Conclusion:

In general, we can conclude that isotropically spin-correlated particles, can only occur in pairs. This suggests a “spin-coupling principle”. It follows that three or more such correlated particles cannot exist and in particular, the structure of spin- $\frac{3}{2}$  baryons cannot be composed of three isotropically correlated quarks, without giving rise to a mathematical contradiction. However, their statistical structure may be explained in terms of the coupling principle, without any recourse to the concept of color. Finally, we note that the Pauli exclusion principle can be derived from the “spin coupling principle”.

## 6 Appendix – Alternate Proof of Pauli Exclusion Principle

**Theorem 2:** *A sufficient condition for a system of  $n$  indistinguishable particles to exhibit fermi-dirac statistics is that it contain spin-coupled particles.*

**Proof:** Let  $\psi[\lambda_i, \lambda_j]$  represent the coupled state. From the coupling principle we get

$$0 \leq P(\lambda_1, \dots, \lambda_i \dots \lambda_i \dots \lambda_n) \leq P(\lambda_i = \lambda_j) \leq P(s_i = s_j) = 0$$

Therefore

$$\psi[\lambda_1, \dots, \lambda_i \dots \lambda_i \dots \lambda_n] = 0 \quad (1)$$

The general form of the wave function is given by

$$\begin{aligned} \psi[\lambda_1, \dots, \lambda_n] &= \sum_P \sigma_P c_p \psi_1(\lambda_i) \otimes \psi_2(\lambda_j) \otimes \dots \otimes \psi(\lambda_n) \\ &= \sum_P \sigma_P d_p \psi_1(\lambda_j) \otimes \psi_2(\lambda_i) \dots \otimes \psi(\lambda_n) \end{aligned}$$

where  $c_P$  and  $d_P$  are constants. Indistinguishability implies that  $c_P^2 = d_P^2 = \frac{1}{n!}$ , for each  $P$ . Let  $c = \frac{1}{\sqrt{n!}}$ . If we now put  $\lambda_i = \lambda_j$  then we get from equation (1) that

$$0 = \sum_P (c_P + d_P) \sigma_P \psi_1(\lambda_i) \otimes \psi_2(\lambda_i) \otimes \dots \otimes \psi_n(\lambda_n)$$

Therefore,  $c_P = -d_P$  for each  $P$  by the linear independence of the eigenfunctions. It follows,

$$\begin{aligned} &\psi[\lambda_1, \dots, \lambda_n] \\ &= (c/2) \sum_P \sigma_P (\psi_1(\lambda_i) \otimes \psi_2(\lambda_j) \otimes \dots \otimes \psi_n(\lambda_n) - \psi_1(\lambda_j) \otimes \psi_2(\lambda_i) \otimes \dots \otimes \psi_n(\lambda_n)) \\ &= \sqrt{n!} \psi_1(\lambda_1) \wedge \dots \wedge \psi_n(\lambda_n). \end{aligned}$$

This obeys the fermi-dirac statistics and the result follows. QED

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